A Stochastic Economic Model using Modified Nonhomogeneous Poisson Process with a Birth and Death Diffusion Intensity Rate and External Jump Processes

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Abstract

One of the important functions of the economists is to provide information on the future trend of economic developments, which is important to plan for human activities. Today, economists are interested in describing phenomena in theoretical models involving economic structure by considering the stochastic analogs of classical differences and differential equations. In this paper a description of a stochastic economic model using a modified nonhomogeneous Poisson process is considered. More specifically, the nonhomogeneous Poisson process is derived using an intensity rate follow a birth and death diffusion process with random external jump process. The mean and the variance approximation as well as the predicted and the simulated sample path of such a stochastic economic process are also obtained. Numerical examples for the case of no jumps as well as the case of the occurrence of jump process that follow a uniform and exponential distributions are considered.


1. Introduction

Poisson Processes either homogeneous or Nonhomogeneous play an important and a fundamental rule in theory and applications that embrace queuing and inventory models, population growth, engineering systems, maintenance theory, economic development, etc. This paper shows how decision makers’ concerns about model specification can affect the trend of economic developments. The importance of using this new approach of stochastic economic process using a modified nonhomogeneous Poisson process in economic models because this model is from the type of continuous – time models which is different from what we have already known from the discrete-type models and this kind of models is not widely used in various economic models.

David (1997) studies a model in which production is linear in the capital stocks with technology stocks that have hidden growth rates. Veronesi (1999) studies a permanent income model with a riskless linear technology. Dividends are modeled as an additional consumption endowment. Hidden information was introduced into asset pricing models by Detemple (1986), who considers a production economy with Gaussian unobserved variables.
Al-Eideh and Hasan (2002) have considered growth price models under random environment using a solution of stochastic differential equation of the logistic price model as well as the logistic price models with random external jump process. They derived the steady state probability and the time dependent probability functions. Also, the mean and the variance as well as the sample path of such a process are considered.

During this past decade there has been increasing effort to describe various facts of dynamic economic interactions with the help of stochastic differential processes. Thus stochastic differential processes provide a mechanism to incorporate the influences associated with randomness, uncertainties, and risk factors operating with respect to various economic units (stock prices, labor force, technology variables, etc.)

Therefore, the techniques of stochastic processes become relevant in pursuing quantitative studies in economics. Stochastic modeling techniques are not only enable us to obtain reliable estimates of certain useful economical parameters but also provides indispensable tools for estimating parameters which are often associated with a high degree of non-sampling errors. Thus, it is proposed to introduce a study of economic structure using the techniques of stochastic processes.


In particular, Mukherjee et al. (1964), He used a generalized form of Poisson process for explanation of economic development with applications to India data. Also, Gusak (1999) studied the Ruin problems for Nonhomogeneous semi continuous integer-valued process.

Numerous researchers have worked on studying various economic units from different points of view. For example, Aase and Guttrop (1987) studied the role of security prices allocative in capital market, they present stochastic models for the relative security prices and shows how to estimate these random processes based on historical price data. The models they suggest may have continuous components as well as discrete jumps at random time points. Also, two classical applications are Metron (1971) and Black and Scholes (1973). New references include Harrison and Pliska (1981) and Aase (1984). Whereas the first two works only study processes with continuous sample paths, the other two allow for jumps in the paths as well. In other words, the processes have sample paths that are continuous from the right and have left hand limits (in fact, these processes are semi-martingales; for general theory of semi-martingales, see e.g. Kabanov et al., 1979 sec. 2).

In this paper, we present a new stochastic economic model using a modified nonhomogeneous Poisson process. More specifically, the nonhomogeneous Poisson process using an intensity rate follow a birth and death diffusion process with random external jump process is studied. The mean and the variance approximation as well as the sample path of such a process are also obtained. The mean and the variance as well as the predicted and the simulated sample path of such a modified nonhomogeneous Poisson process are also obtained. Numerical examples for the case of no jumps as well as the case of the occurrence of jump process that follow a uniform and exponential distributions are considered.

The objective of this research is to identify critical knowledge types required by developers of economic techniques through building a modified Nonhomogeneous Poisson process model to be applied to the trends of economic development such as the national income. This topic has never been examined before as far as I know. The results should be very useful and will benefit the economists and others to study the behaviour of economic development trends through different applications.
2. The Modified Nonhomogeneous Poisson Model using a Birth and Death Diffusion Intensity Rate Process with External Jump Process

Let \( x \) be a certain event and contemplate its possible change in the small time interval between \( t \) and \( t + \Delta t \). The probability of a transition from \( x \) to \( x + 1 \) is

\[
\lambda_i(\Delta t) + o(\Delta t)
\]

(1)

The probability of no change is \( 1 - \lambda_i(\Delta t) + o(\Delta t) \); all other possible transitions are of order \( o(\Delta t) \). Note that we take \( \lambda_i \) such that \( \lambda_i = \frac{dL(t)}{dt} \) where \( L(t) \) represents the birth and death diffusion intensity rate with external jump process.

Consider the intensity rate process \( \{L(t); t \geq 0\} \) in which the diffusion coefficient \( a \) and the drift coefficient \( b \) are both proportional to \( L(t) \) at time \( t \). The diffusion process is assumed to be interrupted by external effects occurring at a constant rate \( c \) and having magnitudes with distribution \( H(\cdot) \). Then \( \{L(t); t \geq 0\} \) is a Markov process with State Space \( S = [0, \infty) \) and can be regarded as a solution of the stochastic differential equation

\[
dL(t) = bL(t)dt + aL(t)dW(t) - L(t^-)dZ(t)
\]

(2)

Here \( \{W(t)\} \) is a Wiener process with mean zero and variance \( \sigma^2 t \). Also, \( \{Z(t)\} \) is a compound Poisson process

\[
Z(t) = \sum_{i=1}^{N(t)} Y_i
\]

(3)

Here \( \{N(t)\} \) is a Poisson process with mean rate \( c \), where \( c \) is the external jump rate, and \( Y_1, Y_2, \ldots \), are independent and identically distributed random variables with distribution function \( H(\cdot) \), with mean \( \mu = E(Y_1) \) and variance \( \nu^2 = Var(Y_1) \). Note that the moments of \( Z(t) \) can be determined from the random sums formulas, and are

\[
E[Z(t)] = c \mu t
\]

(4)

and

\[
Var[Z(t)] = c(\nu^2 + \mu^2) t
\]

(5)

(cf. Taylor and Karlin (1984), pp. 55, 201)

We obtain the probability \( P_x(t) \) that \( x \) has a given value at the time \( t \), the following equation

\[
P_x(t + \Delta t) = \lambda_i(\Delta t)P_{x-1}(t)[1 - \lambda_i(\Delta t)]P_x(t)
\]

(6)

If we go to the limit, which is for \( \Delta t \rightarrow 0 \), we obtain the differential – difference equation

\[
\frac{dP_x(t)}{dt} = \lambda_i P_{x-1}(t) - \lambda_i P_x(t)
\]

(7)
We introduce the generating function

\[ F(s,t) = \sum_{x=0}^{\infty} P_x(t)s^x \]  

(8)

And obtain from equation (7) the differential equation

\[ \frac{dF(s,t)}{dt} = \lambda_t(s-t)F(s,t) \]  

(9)

Integration both sides of (9) give

\[ F(s,t) = k \exp\left\{ (s-1)\int_0^t \lambda_s dt \right\} \]  

(10)

and from the condition that all times \( t \) the sum of the probabilities is one. i.e.

\[ F(1,t) = \sum_{x=0}^{\infty} P_x(t) = 1 \]  

(11)

We obtain \( k = 1 \), so that finally the generating function is

\[ F(s,t) = \exp\left\{ (s-1)\int_0^t \lambda_s dt \right\} \]  

(12)

This is the generating function of the Nonhomogeneous Poisson distribution for the probabilities

\[ P_x(t) = \frac{e^{-L(t)}(L(t))^x}{x!}; \quad x = 0,1,2,\ldots, \]  

(13)

such that \( L(t) \) is the solution of the SDE in equation (2) above. i.e.

\[ L(t) = \int_0^t \lambda_s dt \]  

(14)

Now from equation (2) we get,

\[ \frac{dL(t)}{L(t)} = bdt + adW(t) - dZ(t) \]  

(15)

Thus, the solution of the stochastic differential equation in (15) is given by

\[ L(t) = L(0) \exp\{bt + aW(t) - Z(t)\} \]  

(16)

where \( L(0) \) is the initial intensity rate at time zero. This is known as the mean and the variance of this distribution.

Now, in case of stochastic economic processes, we might construct a process, by defining
\[ \Lambda(t) = \int_0^t L(t) dt \] (17)

for some measure of economic development, for instance national income. Then \( y \) will be a modified process on \( s = \Lambda(t) \) where \( \Delta s = \lambda_y(\Delta t) + o(\Delta t) \); using the same technique as above we find that this is a modified nonhomogeneous Poisson process with probability function:

\[ P_y(t) = \frac{e^{-\Lambda(t)}(\Lambda(t))^y}{y!}; \quad y = 0, 1, 2, \ldots, \] (18)

and using Taylor and Karlin (1984, pp. 177) and Al-Eideh and Al-Hussainan (2002) and after some algebraic manipulations, it is easily shown that

\[ \Lambda(t) = \int_0^t L(s) ds = \int L(0) \exp\{bs + aW(s) - Z(t)\} ds \\
= \frac{2(1-b)}{2a + a^2 - b^2} L(0) \exp\{bt + aW(t) - Z(t)\} \] (19)

where \( L(0) \) is the initial intensity rate of the nonhomogeneous Poisson process at time zero.

3. Mean and Variance Approximation of the Modified Nonhomogeneous Poisson Model \( Y(t) \) Using the Birth and Death Diffusion Intensity Rate with External Jump Processes

In this section, the mean and the variance approximation for the stochastic economic process \( Y(t) \) using the modified nonhomogeneous Poisson intensity rate process \( \Lambda(t) \) with external jump process \( H(t) \).

Let \( M_1(t) = E[Y(t)] \) and \( V(t) = V[Y(t)] \) be the mean and the variance of \( Y(t) \) respectively.

Using the results of finding the moment approximation of a birth and death diffusion process with constant rate jump process (cf. Al-Eideh (2001)), it is easily shown that

\[ E[Y(t)] = \Lambda(0) \frac{2(1-b)}{2a + a^2 - b^2} \exp\left\{ b + \frac{1}{2} a \sigma^2 t \right\} \cdot \left\{ 1 - c \mu \tau + \frac{1}{2} c(v^2 + \mu^2) t \right\} \] (20)

And

\[ E[Y^2(t)] = \left( \Lambda(0) \right)^2 \left( \frac{2(1-b)}{2a + a^2 - b^2} \right)^2 \exp\left\{ 2b + 2a^2 \sigma^2 \right\} \left[ 1 - 2c \mu \tau + 2c(v^2 + \mu^2) t \right] \] (21)

Therefore, the variance of \( Y(t) \) is then given by
\[ V[Y(t)] \approx (\Lambda(0))^2 \left( \frac{2(1-b)}{2a + a^2 - b^2} \right)^2 \exp \left\{ \left[ \frac{2b + 2a^2 \sigma^2}{2a + a^2 - b^2} \right] t \right\} \]
\[ \cdot \left\{ e^{\sigma^2 \gamma} \left( 1 - 2c \mu t + 2c(v^2 + \mu^2)t \right) - \left[ 1 + c \mu t \right]^2 + \left( v^2 + \mu^2 \right) \left( \frac{1}{4} c^2 t^2 + ct - c^2 \mu^2 \right) \right\} \]  

where \( \Lambda(0) \) is the initial intensity rate of the modified nonhomogeneous Poisson process \( Y(t) \) at time zero.

4. Predicted and Simulated Modified Nonhomogeneous Poisson Model \( Y(t) \)

In this section, we will obtain the predicted and the simulated sample path of the modified nonhomogeneous Poisson process \( Y(t) \) using the birth and death diffusion process \( \Lambda(t) \) with external jump process \( H(\cdot) \).

Assuming \( M_1(t_n - t_{n-1}) \) be the one-step predicted model of \( Y(t) \), then \( M_1(t_n - t_{n-1}) \) can be written as

\[ M_1(t_n - t_{n-1}) = \Lambda(0) + \frac{2(1-b)}{2a + a^2 - b^2} \exp \left\{ \left( b + \frac{1}{2} a \sigma^2 \right) (t_n - t_{n-1}) \right\} \]
\[ \cdot \left\{ 1 - c \mu (t_n - t_{n-1}) + \frac{1}{2} c(v^2 + \mu^2)(t_n - t_{n-1}) \right\} \]

Using the technique of Ross (1998, pp. 464), it is easily shown that the simulation of the modified nonhomogeneous Poisson Process \( Y(t) \) is given by

\[ Y(t) = \max \left\{ n : \sum_{i=1}^{n} - \log U_i \leq \Lambda(t) \right\} \]  

where \( U_1, U_2, U_3, \ldots \) be independent uniform random variables on \([0,1]\).

For simulation of the stochastic economic process \( Y(t) \) we used the following discrete approximation.

For integer values \( k = 1,2,3,\ldots \), and \( n = 1,2,3,\ldots \), the intensity rate diffusion process \( \Lambda(t) \) with external jump process \( H(\cdot) \) can be simulated by

\[ \Lambda_n^* \left( \frac{k+1}{n} \right) = \Lambda_n^* \left( \frac{k}{n} \right) + \frac{2(1-b)}{2a + a^2 - b^2} \left( \frac{b}{n} \Lambda_n^* \left( \frac{k}{n} \right) + \frac{a}{n} \Lambda_n^* \left( \frac{k}{n} \right) \cdot Z_{k+1} - \Lambda_n^* \left( \frac{k}{n} \right) \right) \Delta C \left( \frac{k}{n} \right) \]

where \( \{Z(k)\} \) is an independent sequence of standard normal random variables and \( \Delta C \left( \frac{k}{n} \right) \); \( k = 1,2,\ldots \) are independent and identically distributed with

\[ P \left( \Delta C \left( \frac{k}{n} \right) = 1 \right) = \frac{c}{n} \]
\[ P \left( \Delta C \left( \frac{k}{n} \right) = 0 \right) = 1 - \frac{c}{n} \]
and \( J\left(\frac{1}{n}\right), J\left(\frac{2}{n}\right), \ldots \) are independent and identically distributed with distribution \( H(\cdot) \).

For each set of positive integers \( k, t_1, \ldots, t_k \), the sequence of random vectors \((\Lambda^*_n(t_1), \ldots, \Lambda^*_n(t_k))'\) converges in distribution to \((\Lambda^*_n(t_1), \ldots, \Lambda^*_n(t_k))'\).

Thus, the simulated stochastic economic model \( Y(t) \) is given by

\[
Y^*_n\left(\frac{k+1}{n}\right) = Y^*_n\left(\frac{k}{n}\right) + \max\left\{n : \sum_{i=1}^{n} -\log U\left(\frac{i}{n}\right) \leq \Lambda^*_n\left(\frac{k+1}{n}\right)\right\}
\]

(26)

where \( \Lambda^*_n\left(\frac{k+1}{n}\right) \) in defined in equation (25) and \( U\left(\frac{1}{n}\right), U\left(\frac{2}{n}\right), \ldots \) are independent and identically distributed with Uniform distribution on \((0,1)\).

Note that for each set of positive integers \( k \), the sequence of random vectors \( \left(Y^*_n(1), \ldots, Y^*_n(k)\right)'\) converges in distribution to \( \left(Y(1), \ldots, Y(k)\right)'\).

**Numerical Example**

Consider as an example the following sample paths of the above model \( Y(t) \) in section (4) that represents the annual national income of an anonymous country in Millions US dollars when \( Y(0) = 700, b = 0.02, a = 2, n = 20, \) and \( c = 1 \) for the following cases:

**Case 1**

In this case we consider the sample path to the modified nonhomogeneous Poisson stochastic economic model \( Y(t) \) using the stochastic birth and death diffusion intensity rate process \( \Lambda(t) \) with no external jump process, note in this case the jump rate \( c = 0 \). Figure 1 and Figure 2 represent this case for \( \Lambda(t) \) and \( Y(t) \) respectively.

**Figure 1:** The Stochastic Intensity Rate Process with no External Jump Process
Figure 2: The Associated Modified Nonhomogeneous Poisson Stochastic Economic Model

Case 2

In this case we consider the sample path to the modified nonhomogeneous Poisson stochastic economic model \( Y(t) \) using the stochastic birth and death diffusion intensity rate process \( \Lambda(t) \) with Uniform external jump process, note in this case the jump rate \( c = 1 \). For simplicity we take \( H(\cdot) \) to be uniform on \([0,1]\). Thus

\[
dH(y) = 1, \quad 0 \leq y \leq 1
\]  

(27)

Note that \( H(y) \) is independent of \( y \) with mean \( \mu = \frac{1}{2} \), and variance \( \nu^2 = \frac{1}{12} \). Figure 3 and Figure 4 represent this case for \( \Lambda(t) \) and \( Y(t) \) respectively.

Figure 3: The Stochastic Intensity Rate Process with Uniform Jump Process
Figure 4: The Associated Modified Nonhomogeneous Poisson Stochastic Economic Model

Case 3

In this case we consider the sample path to the modified nonhomogeneous Poisson stochastic economic model $Y(t)$ using the stochastic birth and death diffusion intensity rate process $\Lambda(t)$ with Exponential external jump process, note in this case the jump rate $c=1$. For simplicity we take $H(\cdot)$ to be exponential with mean 1. Thus

$$dH(y) = e^{-y}, \quad y > 0$$

Note that $H(y)$ depends on $y$ with mean $\mu = 1$, and variance $\nu^2 = 1$. Figure 5 and Figure 6 represent this case for $\Lambda(t)$ and $Y(t)$ respectively.

Figure 5: The Stochastic Intensity Rate Process with Exponential Jump Process
Looking to the above figures, we see the difference between these figures. In addition, the difference between the uniform jump and the exponential jump is noted in the stochastic intensity rate processes and this show the difference between the jump processes if they are dependent or independent of the intensity rates and finally this difference affects the modified nonhomogeneous Poisson stochastic economic models. Any way the figures are reasonable and suggested to be used in the modeling purposes for some trend of economic developments.

**Conclusion**

In conclusion this study provides a methodology for studying the behavior of the trend of economic developments such as national incomes, etc. More specifically, the study departs from the traditional before and after regression techniques and the time series analysis and developed a stochastic model that explicitly accounts for the variations and volatilities in economic development trends using a modified nonhomogeneous Poisson processes with birth and death diffusion intensity rate process subject to randomly occurring external jump processes, especially the uniform on $[0, 1]$ and exponential with mean 1 processes. Ideally, a large class of external jump processes with general jump rate could be tackled in future researches. Also some inference problems could be done for this model.

In terms of future research, this methodology could be applied not only in trend of economic developments but on all aspects of economics and operations research problems.

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**References**


